THE

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SO-CALLED CASES OF FAILURE IN THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS.

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(*Concluded.*)

WE turn now to the question of the generality of the solution of equation (2) obtainable by means of equation (3), that is, to the problem of the construction of integrals of the equation

$$G(y)=0. \quad\text{..(17)}$$

In passing, we remark that it is shewn by the last example that integrals of equation (2) deduced from different functions satisfying the conditions of equation (3) may differ, so that a number of independent functions satisfying equations of the form of (3) may lead to an equivalent number of independent solutions of (2), or may lead to any smaller number other than zero.

Again, any arbitrary constants occurring in the function $F(x, p_1, p_2, \dots p_r)$ may be regarded as arbitrary functions of any or all of the parameters $p_1, p_2, \dots p_r$ without affecting the relation between the equations (2), (3), the only conditions which may have to be imposed being negative conditions, preventing them or certain combinations of them from becoming zero or infinite for the special values $a_1, a_2, \dots a_r$ of the parameters; it is easy to prove, from the way in which the arbitrary constants must enter into the function F, that the derived solution of equation (2) contains the same number of arbitrary constants as the given function.

Considerations of a different kind appear when we relate the solution of equation (17), or in other words the discovery of terms in the complementary function of equation (2), to the given result (3).

The first conclusion from the equation (3) in this connection is obvious: if for any set of values $b_{t_1}, b_{t_2}, \dots b_{t_r}$ of $p_1, p_2, \dots p_r$ the function $h(p_1, p_2, \dots p_r)$ is zero, the function $F(x, b_{t_1}, b_{t_2}, \dots b_{t_r})$ is a solution of equation (17). We may call such a set a zero-set of the function h. If there is only one parameter p, the values of p for which $h(p)$ vanishes are fixed numbers, the roots of the equation $h(p)=0$, but if there are more parameters than one, we can,

as a rule, fix arbitrarily all but one of the parameters and determine the remaining one by the condition that the function vanishes; this remaining parameter is a function of the parameters which we consider arbitrary, and if one of the latter varies, the dependent parameter varies also. In fact, if there are more parameters than one, the values forming a zero-set are not determinate but are capable of continuous variation; for this reason the case of a single parameter is much the easier to discuss, and we commence with it.

The problem now before us is this: given a homogeneous linear differential operator G involving the independent variable x in any way whatever, and a function $F(x, p)$ involving a parameter p and known to be such that

$$G\{F(x, p)\}=h(p)f(x, p), \quad \text{......................(18)}$$

to construct solutions of the differential equation

$$G(y)=0.$$

Without loss of generality we may suppose that $f(x, p)$ does not vanish or become infinite irrespectively of x for any finite value of p; on account of the form of G we could make the same assumption as to $F(x, p)$, but it is in fact inconvenient to assume that $F(x, p)$ cannot vanish identically for certain values of p; so with regard to $F(x, p)$ we assume only that it does not become identically infinite for any finite value of p.

From equation (18) and the restrictions imposed on $f(x, p)$ it follows that the necessary and sufficient condition for $F(x, p)$ to be a solution of $G(y)=0$ is that p should be a root of the equation $h(p)=0$, or in other words a zero of the function $h(p)$. Every zero of $h(p)$ will give rise to a solution of the equation, but this solution may be only the trivial one $y=0$; the zeros of $h(p)$ which lead to integrals other than $y=0$ may be distinguished as efficient zeros.

Again, from (18) we have, for all values of p, the equation

$$G\{\partial F/\partial p\}=(dh/dp)f(x, p)+h(p)(\partial f/\partial p), \quad \text{..................(19)}$$

which is only the particular case, for one parameter, of an equation already written out and used. But whereas before we were anxious that this equation should *not* reduce to the form $G(y)=0$, now it is precisely of this last equation that we are trying to find solutions. Even with the conditions already imposed, $\partial f/\partial p$ may vanish identically for a particular value of p which is also a zero of dh/dp, but such a case is quite special. And the only other case in which the right-hand side of equation (19) can be seen from its form to vanish is that in which p has a value causing both $h(p)$ and dh/dp to vanish, or in other words that in which $h(p)$ has a multiple zero. It is worth while to notice that if b is a simple zero of $h(p)$, the function $G\{\partial F/\partial b\}$ cannot in any circumstances vanish identically. We may differentiate equation (19) with respect to p and argue in the same way from the identity which results that $G\{\partial^2 F/\partial p^2\}$ certainly vanishes when the value of p is a triple zero of $h(p)$ and certainly does not vanish when the value of p is a double but not a triple zero of this function, and that the only other values of p which the general form of the identity shews to furnish solutions of equation (17) are values which are simultaneously double zeros of $h(p)$ and single zeros, for all values of x, of $\partial f/\partial p$, and values which are simultaneously single zeros of $h(p)$ and double zeros, identically with respect to x, of $\partial f/\partial p$. And, in general, if $h(p)$ has zeros $b_1, b_2, \ldots$ of multiplicities $q_1, q_2, \ldots$, as many of the $\Sigma(q_k)$ functions

$$F(x, b_1),\ \partial F/\partial b_1, \ldots \partial^{q_1-1}F/\partial b_1^{q_1-1};\quad F(x, b_2),\ \partial F/\partial b_2, \ldots \partial^{q_2-1}F/\partial b_2^{q_2-1};\ \ldots,$$

as are not identically zero are proper solutions of the differential equation $G(y)=0$.

For example, from the equation (4) it follows that the values which the

functions e^{px}, $\partial e^{px}/\partial p$ assume when $p=1$, that is, the functions e^x, xe^x, are solutions of the equation

$$(D-1)^2y=0,$$

and incidentally it follows that the function x^2e^x is not a solution of the equation.

The functions $f(x, p)$, $F(x, p)$ are just as likely to tend to finite forms as p tends to infinity as they are to take finite forms when p takes a finite value, and if we make the general assumption that neither the function $f(x, p)$, nor the function $F(x, p)$, nor any one of the derivatives of the latter function with respect to p, becomes identically infinite as p takes any constant value, finite or infinite, we can assert that the number of solutions of the equation $G(y)=0$ which a knowledge of a number of the zeros of the function $h(p)$ enables us to obtain, is equal to the order of $h(p)$ in the known zeros, that is, to the sum of the multiplicities of these zeros, and if the knowledge of the zeros of $h(p)$ is complete, a number of solutions equal to the total order of this function can be found.

To see the importance of this result, we may consider the operator G as obtained from an operator G_u involving a number s, not necessarily greater than one, of arbitrary parameters $u_1, u_2, \dots u_s$ by the giving of special values $c_1, c_2, \dots c_s$ to these parameters, and we may suppose that a result of the form

$$G_u\{F_u(x, p)\}=h_u(p)f_u(x, p)$$

is known, the various functions involving, in addition to the variables x, p, the parameters $u_1, u_2, \dots u_s$, and that these functions reduce to the functions we have been considering when the parameters take the values $c_1, c_2, \dots c_s$. Then although a variation of these parameters will usually change the positions of the zeros of the function $h_u(p)$, causing coalescence and separation, it will not as a rule cause any change in the total order of this function, and when the order of the function is unchanged so also is its value for the determination of solutions of the differential equation $G_u(y)=0$.

For example, although the particular solutions e^{lx}, e^{mx} of the equation

$$(D-l)(D-m)y=0$$

which are suggested by the result

$$(D-l)(D-m)e^{px}=\{(p-l)(p-m)\}e^{px}$$

coalesce when l and m are equal, the corresponding *identity*

$$(D-l)^2e^{px}=(p-l)^2e^{px}$$

still has a quadratic function of p for a factor in the right-hand side, and is still effective for suggesting two solutions of the equation

$$(D-l)^2y=0,$$

namely e^{lx} and xe^{lx}.

But it is to be observed that neither in this particular case nor in general is it necessary or even useful to have acquaintance with the solution of a more general equation under which a proposed equation is included. Unless the general equation happens to be such as to suggest to us a function $F(x, p)$ leading to a function $h(p)$ radically different from any such function already known, it is unlikely that by attacking the general equation we shall make any advance in solving the particular one.

The best illustration of this part of the theory is to be found in the method devised by Frobenius for constructing regular integrals of homogeneous linear equations. A function $F(x, p)$ is formed such that

$$G\{F(x, p)\}=h(p)x^p,$$

this function not becoming infinite identically for any finite value of p and $h(p)$ being a polynomial in p; the function $F(x, p)$ is constructed as a power series in x whose coefficients are rational functions of p, and it is not always convenient to prevent these coefficients from having a common

factor in the numerator, though they must not have one in the denominator; for some of the zeros of $h(p)$ the function $F(x, p)$ may vanish identically, and it has been proved by Fuchs and by Frobenius (for references, a proof, and examples, see Forsyth's *Theory of Differential Equations*, vol. iv.; examples are given also in Forsyth's *Treatise on Differential Equations*, pp. 243-258 (4th ed., 1914)) that the remaining zeros, which are the efficient zeros of the polynomial, are always sufficiently numerous to lead to all the regular integrals of the equation $G(y)=0$. The precise result is that if the efficient zeros are divided into groups so that the real part of each zero differs from the real part of every zero in its own group by an integer and from the real part of every zero in every other group by an amount which is not an exact integer, and if the members of each group are so arranged that the real part of each is greater than the real part of the zero which follows it in its own group, then the multiplicity q_{kl} of b_{kl}, the lth member of the kth group, is greater than the multiplicity $q_{k,l-1}$ of the zero $b_{k,l-1}$ which precedes it in its own group, and if we denote the value of $\partial^m F/\partial p^m$ when p has the value b_{kl} by $F_{kl}(m)$, the set of functions obtained by forming from each zero b_{kl} the $q_{kl}-q_{k,l-1}$ functions

$$F_{kl}(q_{k,l-1}),\quad F_{kl}(q_{k,l-1}+1),\quad F_{kl}(q_{k,l-1}+2), \dots F_{kl}(q_{kl}-1)$$

is a complete set of linearly independent regular integrals of the equation $G(y)=0$, the functions associated with the first member b_{k1} of the group being the values of F itself and of its first $q_{k1}-1$ derivatives, that is, being

$$F_{k1}(0),\quad F_{k1}(1), \dots F_{k1}(q_{k1}-1).$$

It will be observed that the number of functions thus obtained from the first l members of a group is equal to the multiplicity of the lth member itself, and that from this lth member alone we can obtain this number of integrals, namely $F_{kl}(0)$, $F_{kl}(1)$, ... $F_{kl}(q_{kl}-1)$, but examples can be found to shew that the first $q_{k,l-1}$ members of this last set of functions are not necessarily linearly independent and indeed that some or all of them may vanish identically. The total number of integrals obtainable is the sum of the multiplicities of the last members of the various groups, and this is equal to or less than the order of the equation according as it is all or only some of the integrals of the equation which are regular.

In conclusion, we must return to the construction of integrals of equation (17) on the basis of an equation of the form (3) involving more than one parameter. There are two possibilities, illustrated by the examples already given, which might be thought distinct: the vanishing of the function $h(p_1, p_2, \dots p_r)$ may be due solely to the taking of a particular value b_{tk} by one of the parameters p_k, independently of the other parameters, or it may be due to the satisfying of a relation really determining one parameter as a function of the others. But there is nothing vital in this distinction; in either case $r-1$ of the parameters remain arbitrary, and the function $F(x, p_1, p_2, \dots p_r)$ is a function from which in general not all the arbitrary elements will disappear in virtue of the one equation $h(p_1, p_2, \dots p_r)=0$. It is natural to regard as the general case that in which, after substitution in $F(x, p_1, p_2, \dots p_r)$ for one of the parameters in terms of the others in accordance with the equation $h(p_1, p_2, \dots p_r)=0$, the function remains a function of x and of $r-1$ parameters; then it is a solution of the differential equation (17) involving $r-1$ constants, so that $r-1$ cannot be greater than n, the order of the equation, and if $r-1$ is equal to n, the solution is general. But it is equally conceivable for $F(x, p_1, p_2, \dots p_r)$ to be of the form

$$H(x)+h(p_1, p_2, \dots p_r)K(x, p_1, p_2, \dots p_r),$$

in which case equation (3) assists us to only a particular solution of equation (17); any intermediate case is possible, and in fact the real difference

between various functions $h(p_1, p_2, \ldots p_r)$ from the point of view of utility in this connection resides in the generality remaining in $F(x, b_{t_1}, b_{t_2}, \ldots b_{t_r})$, where $b_{t_1}, b_{t_2}, \ldots b_{t_r}$ is a zero-set of the function $h(p_1, p_2, \ldots p_r)$. And what it is important to observe is that if a zero-set varies continuously it may yield always the same integral of the equation (17), or it may yield any number of integrals which are analytically distinguishable, but it cannot yield more than $r-2$ functions which are linearly independent solutions of the differential equation.

Multiplicity of a zero-set is the last point to which we have to refer. If the function h together with one of its derivatives of order m and all those derivatives of lower orders which are subsidiary to this particular derivative vanishes for the values $b_{t_1}, b_{t_2}, \ldots b_{t_r}$ of the parameters, then the function F and all its corresponding derivatives including that of the mth order, assume for this set of values of the parameters forms which are integrals of equation (17).

Again we may illustrate from equation (10) how the fact of having more than one parameter may be utilised. To obtain more than one solution of the equation

$$(D^2-2D+2)y=0$$

from the one-parameter results (14), (15), (16) we have to introduce complex numbers, and then the factors of $F(x, p)$ exchange parts in a curious fashion. The roots of $h(p)$ are $1, i, -i$, all simple, the function $F(x, 1)$ is real, but the functions $F(x, i), F(x, -i)$ are complex, and

$$(1+i)F(x, 1)+iF(x, i)+F(x, -i)=0.$$

If we take as our basis the two-parameter results (11), (12), (13) we have a zero-set of $h(p, q)$ which varies continuously, and $-2(p-1)qe^{(p+iq)x}$ is an integral for all values of p, q satisfying the equation $h(p, q)=0$. The only pairs of real values of p, q such that $h(p, q)=0$ are $p=1, q=\pm 1$; these values cause the integral just found to vanish identically, so that a certain trivial amount of justification has to be made of the statement that the finite functions $e^{(1+i)x}, e^{(1-i)x}$ are in fact integrals, and even then the functions are complex. Trouble is avoided by remarking that the sets $(1, 1)$ and $(1, -1)$ are double zero-sets of $h(p, q)$ with respect to both the parameters p, q, so that four particular integrals (a number equal to the degree of $h(p, q)$ in the two parameters) are given as the values of $\partial F/\partial p$ and $\partial F/\partial q$ for the two sets. The two values of $\partial F/\partial p$ are $\mp 2e^x \cos x$ and are not linearly independent, and the two values of $\partial F/\partial q$ are $\mp 2e^x \sin x$, and these though not linearly independent of each other are linearly independent of those already found. The equation being of order two, its complete solution is now in our hands, but we may notice that the values of $\partial^2 F/\partial p\, \partial q$ for the two zero-sets also are particular integrals; as a matter of fact they coincide, both of them being $-2e^x \cos x$.

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APPROXIMATIONS TO $\sqrt[n]{1+x}$, WHERE n IS AN INTEGER AND $0<x<1$.

BY J. M. CHILD.

IF we seek approximations to $\sqrt[n]{1+x}$, in the form

$$\frac{1+a_1x+a_2x^2+\ldots+a_rx^r}{1+b_1x+b_2x^2+\ldots+b_rx^r},$$

of such a nature that the fraction gives, on performing the division, an infinite series whose first $2r+1$ terms are identical with the first $2r+1$ terms

of the Binomial Expansion for $\sqrt[n]{1+x}$; then we shall obtain $2r$ equations of the form :

$$a_1=b_1+b_0\frac{1}{n},$$

$$a_2=b_2+b_1\frac{1}{n}+b_0\frac{1}{\underline{|2}}\frac{1}{n}\left(\frac{1}{n}-1\right),$$

....

$$a_r=b_r+b_{r-1}\frac{1}{n}+b_{r-2}\frac{1}{\underline{|2}}\frac{1}{n}\left(\frac{1}{n}-1\right)+\dots+b_0\frac{1}{\underline{|r}}\frac{1}{n}\left(\frac{1}{n}-1\right)\dots\left(\frac{1}{n}-\overline{r+1}\right),$$

$$0=b_r+b_{r-1}\frac{\underline{|1}}{\underline{|2}}\left(\frac{1}{n}-1\right)+b_{r-2}\frac{\underline{|1}}{\underline{|3}}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)+\dots$$
$$+b_0\frac{\underline{|1}}{\underline{|r+1}}\left(\frac{1}{n}-1\right)\dots\left(\frac{1}{n}-\overline{r+2}\right),$$

$$0=b_r+b_{r-1}\frac{\underline{|2}}{\underline{|3}}\left(\frac{1}{n}-2\right)+b_{r-2}\frac{\underline{|2}}{\underline{|4}}\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)+\dots$$
$$+b_0\frac{\underline{|2}}{\underline{|r+2}}\left(\frac{1}{n}-2\right)\dots\left(\frac{1}{n}-\overline{r+3}\right),$$

....

$$0=b_r+b_{r-1}\frac{\underline{|r}}{\underline{|r+1}}\left(\frac{1}{n}-r\right)+b_{r-2}\frac{\underline{|r}}{\underline{|r+2}}\left(\frac{1}{n}-r\right)\left(\frac{1}{n}-r+1\right)+\dots$$
$$+b_0\frac{\underline{|r}}{\underline{|2r}}\left(\frac{1}{n}-r\right)\dots\left(\frac{1}{n}+2r\right),$$

where b_0 is unity.

For instance,

(i) $r=1$, $a_1=b_1+\frac{1}{n}$, $0=b_1+\frac{1}{2}\left(1-\frac{1}{n}\right)$; $\therefore$ $b_1=\frac{n-1}{2n}$, $a_1=\frac{n+1}{2n}$.

(ii) $r=2$, $a_1=b_1+\frac{1}{n}$, $a_2=b_2+b_1\frac{1}{n}+\frac{1}{2}\frac{1}{n}\left(\frac{1}{n}-1\right)$,

and $$0=b_2+\frac{b_1}{2}\left(\frac{1}{n}-1\right)+\frac{1}{6}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right),$$

$$0=b_2+\frac{b_1}{3}\left(\frac{1}{n}-2\right)+\frac{1}{12}\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right),$$

whence $b_1=\frac{2n-1}{2n}$, $b_2=\frac{(2n-1)(n-1)}{12n^2}$, $a_1=\frac{2n+1}{2n}$, $a_2=\frac{(2n+1)(n+1)}{12n^2}$.

(iii) $r=3$, $b_1=\frac{3n-1}{2n}$, $b_2=\frac{(3n-1)(2n-1)}{10n^2}$, $b_3=\frac{(3n-1)(2n-1)(n-1)}{120n^3}$,

$a_1=\frac{3n+1}{2n}$, $a_2=\frac{(3n+1)(2n+1)}{10n^2}$, $b_3=\frac{(3n+1)(2n+1)(n+1)}{120n^3}$.

The corresponding fractions are

$$\frac{2n+(n+1)x}{2n+(n-1)x},\quad \frac{12n^2+6n(2n+1)x+(2n+1)(n+1)x^2}{12n^2+6n(2n-1)x+(2n-1)(n-1)x^2},$$

$$\frac{120n^3+60n^2(3n+1)x+12n(3n+1)(2n+1)x^2+(3n+1)(2n+1)(n+1)x^3}{120n^3+60n^2(3n-1)x+12n(3n-1)(2n-1)x^2+(3n-1)(2n-1)(n-1)x^3},$$

and so on. The error involved in using any one of these fractions is one in defect, the limits of error being respectively

$$\frac{n^2-1}{12n^3}x^3,\ \frac{(n^2-1)(4n^2-1)}{720n^5}x^5,\ \frac{(n^2-1)(4n^2-1)(9n^2-1)}{100800n^7}x^7,$$

the general term being $\dfrac{(n^2-1)(4n^2-1)(r^2n^2-1)}{2^{2r}.1^2.3^2.5^2.7^2\ldots(2r-1)^2(2r+1)}\left(\dfrac{x}{n}\right)^{2r+1}$.

For instance, it is easy to show that

$$\sqrt[n]{1+x}(2n+\overline{n-1}\,x)$$
$$=2n+(n+1)x+\frac{n^2-1}{6n^2}x^3\left[1-\frac{2n-1}{4n}.2x+\frac{(2n-1)}{4n}\cdot\frac{(3n-1)}{5n}.3x^2-\ldots\right],$$

and each term of the series in the bracket on the left-hand side is less than the preceding, hence the whole series is positive and <1;

$$\therefore\ \sqrt[n]{1+x}-\frac{2n+(n+1)x}{2n+(n-1)x}<\frac{(n^2-1)x^3}{6n^2\{2n+(n-1)x\}}<\frac{(n^2-1)x^3}{12n^3}$$

if $n>1$ and $0<x<1$, for then $2n+(n-1)x$ is positive.

Again $\sqrt[n]{1+x}\{12n^2+6n(2n-1)x+(2n-1)(n-1)x^2\}$

$$=12n^2+6n(2n+1)x+(2n+1)(n+1)x^2$$
$$+\frac{(n^2-1)(4n^2-1)}{120n^3}x^5\left\{1.2-\frac{3n-1}{6n}2.3x+\frac{3n-1}{6n}\cdot\frac{4n-1}{7n}.3.4x^2-\ldots\right\},$$

where the infinite series has alternate signs and each term *after the fourth* is less than the preceding, and therefore the whole series is less than $2-\dfrac{(3n-1)}{n}x+\dfrac{2(3n-1)(4n-1)}{7n^2}x^2$, which is readily shown to be less than

$$\frac{1}{6n^2}[12n^2+6n(2n-1)x+(2n-1)(n-1)x^2];$$

$$\therefore\ \sqrt[n]{1+x}-\frac{12n^2+6n(2n+1)x+(2n+1)(n+1)x^2}{12n^2+6n(2n-1)x+(2n-1)(n-1)x^2}<\frac{(n^2-1)(4n^2-1)}{720n^5}x^5.$$

But it has not yet been shown that the series is positive.

Similarly

$$\sqrt[n]{1+x}\{120n^3+60n^2(3n-1)x+12n(3n-1)(2n-1)x^2+(3n-1)(2n-1)(x-1)x^3\}$$
$$=120n^3+60n^2(3n+1)+12n(3n+1)(2n+1)x^2+(3n+1)(2n+1)(n+1)x^3$$
$$+\frac{(n^2-1)(4n^2-1)(9n^2-1)}{5040n^4}x^2\left\{1.2.3-\frac{4n-1}{8n}.2.3.4x\right.$$
$$\left.+\frac{4n-1}{8n}\cdot\frac{5n-1}{7n}.3.4.5x^2-\ldots\right\}.$$

But here the Algebraic difficulties increase still further, for the infinite series on the left-hand side does not have its terms decreasing until *after the ninth term.*

CONTINUED FRACTION FORM FOR THE BINOMIAL.

The following work is, to a great extent, tentative; and I should be very glad if some one who has made a speciality of continued fractions would correspond with me. It was suggested by the fact that the fractions obtained in the above are successive odd convergents to a continued fraction.

For instance,

$$\begin{array}{ll|r|r|ll}
(1) & 1 & 12n^2+6n(2n+1)x+(2n+1)(n+1)x^2 & 12n^2+6n(2n-1)x+(2n-1)(n-1)x^2 & \dfrac{n}{x} & (2) \\
(3) & \dfrac{2}{n-1} & 12nx+6nx^2 & 6n(n-1)x+(2n-1)(n-1)x^2 & \dfrac{3n(n-1)}{(n+1)x} & (4) \\
 & & & (2n-1)(n-1)x^2 & & \\
(5) & \dfrac{2(n+1)}{(2n-1)(n+1)} & 2(n+1)x^2 & & &
\end{array}$$

$$\therefore\ \text{fraction}=1+\frac{1}{\frac{n}{x}+}\,\frac{1}{\frac{2}{n-1}+}\,\frac{1}{\frac{3n(n-1)}{(n+1)x}+}\,\frac{1}{\frac{2(n+1)}{(2n-1)(n-1)}}.$$

It therefore, from this and other like results, seemed reasonable to conclude that

Expansion for $\sqrt[n]{1+x}$

$$=1+\frac{1}{\frac{n}{x}+}\,\frac{1}{\frac{2}{n-1}+}\,\frac{1}{\frac{3n(n-1)}{(n+1)x}+}\,\frac{1}{\frac{2(n+1)}{(2n-1)(n-1)}+}\,\frac{1}{\frac{5n(n-1)(2n-1)}{(n+1)(2n+1)x}}\ldots\text{ to }\infty.$$

An attempt to use Euler's Theorem (Glaisher's form) for the conversion of the Binomial Series to a continued fraction gives

$$1+\frac{1}{\frac{n}{x}+}\,\frac{\frac{n}{x}}{\frac{2n}{(n-1)x}-1+}\,\frac{\frac{2n}{(n-1)x}}{\frac{3n}{3(n-1)x}-1+}\ldots,$$

which does not seem to reduce to the form suggested by the approximating fractions by any of the ordinary methods.

But $$\sqrt[n]{1+x}-1=\cfrac{x}{n\left\{\cfrac{1}{1-\frac{(n-1)x}{2n}+\frac{(n-1)(2n-1)x^2}{6n^2}-\ldots}\right\}}$$

$$=\frac{x}{n+}\,\cfrac{(n-1)x}{2\left\{\cfrac{1-\frac{n-1}{2n}x+\frac{(n-1)(2n-1)}{6n^2}x^2-\ldots}{1-\frac{2n-1}{3n}x+\frac{(2n-1)(3n-1)}{6n^2}x^2-\ldots}\right\}}$$

$$=\frac{x}{n+}\,\frac{(n-1)x}{2+}\,\frac{(n+1)x}{3n+}\,\frac{(2n-1)x}{2+}\,\frac{(2n+1)x}{5n+}\ldots.$$

The division can be carried on as far as is desired, and the successive quotients follow a general law which is fairly obvious on inspection.

Now this can easily be shown to be the same continued fraction as

$$\frac{1}{\frac{n}{x}+}\,\frac{1}{\frac{2}{n-1}+}\,\frac{1}{\frac{3n(n-1)}{(n+1)x}+}\,\frac{1}{\frac{(2n+1)}{(2n-1)n-1}+}\ldots.$$

The denominators of the successive convergents to the latter are

$$\frac{n}{x},\ \frac{2n+(n-1)x}{(n-1)x},\ \frac{6n^2+2n(2n-1)x}{(n+1)x^2},\ \frac{12n^2+6n(2n-1)x+(2n-1)(n-1)x^2}{(n-1)(2n-1)x^2},\text{ etc.,}$$

which are respectively greater than

$$\ldots,\ \frac{2n}{(n-1)x},\ \frac{6n^2}{(n+1)x^2},\ \frac{12n^2}{(n-1)(2n-1)x^2},$$

$$\frac{60n^3}{(n+1)(2n+1)x^3},\ \frac{120nx^4}{(n-1)(2n-1)(3n-1)x^3},\ \ldots.$$

Hence, if F is the value of the infinite continued fraction, the successive convergents differ from F alternately in excess and defect by quantities less than

$$\ldots,\ \frac{n^2-1}{12n^3}x^3,\ \frac{(n^2-1)(2n-1)}{72n^4}x^4,\ \frac{(n^2-1)(4n^2-1)}{720n^5}x^5,\ \ldots.$$

But the even convergents can be shown* to differ, in defect, from $\sqrt[n]{1+x}$ when $x<1$, by quantities less than these same limits of defect; and this limit can be made as small as we please if $n>1$.

Hence I conclude that if $n>1$, $0<x<1$,

$$\sqrt[n]{1+x}=1+\frac{x}{n+}\,\frac{(n-1)x}{2+}\,\frac{(n+1)x}{3n+}\ldots \text{ to } \infty.$$

If for n we write $\frac{1}{2}$, the continued fraction terminates and becomes

$$1+\frac{x}{\frac{1}{2}+}\,\frac{-\frac{1}{2}x}{2+}\,\frac{\frac{3}{2}x}{\frac{3}{2}}=1+\frac{x}{\frac{1}{2}-}\,\frac{x}{4+2x}=1+\frac{x(4+2x)}{2}=(1+x)^2.$$

Similarly, if $n=\frac{1}{3}$, C.F. $=(1+x)^3$;

$$\text{if } n=-1,\ \text{C.F.}=1+\frac{x}{-1+}\,\frac{-2x}{2}=1+\frac{x}{-1-x}=\frac{1}{1+x}=(1+x)^{-1};$$

$$\text{if } n=-\tfrac{1}{2},\ \text{C.F.}=1+\frac{x}{-\frac{1}{2}+}\,\frac{-\frac{3}{2}x}{2+}\,\frac{\frac{1}{2}x}{-\frac{3}{2}+}\,\frac{-2x}{2}=\frac{1}{1+2x+x^2}=(1+x)^{-2}.$$

Hence, in general, the theorems

$$(1+\mathbf{x})^{\mathbf{m}}=1+\frac{\mathbf{mx}}{1-}\,\frac{(\mathbf{m}-1)\mathbf{x}}{2+}\,\frac{(\mathbf{m}+1)\mathbf{x}}{3-}\,\frac{(2\mathbf{m}-1)\mathbf{x}}{2+}\,\frac{(2\mathbf{m}-1)\mathbf{x}}{5-}\ldots,$$

$$\sqrt[\mathbf{n}]{(1+\mathbf{x})}=1+\frac{\mathbf{x}}{\mathbf{n}+}\,\frac{(\mathbf{n}-1)\mathbf{x}}{2+}\,\frac{(\mathbf{n}+1)\mathbf{x}}{3\mathbf{n}+}\,\frac{(2\mathbf{n}-1)\mathbf{x}}{\mathbf{n}+}\,\frac{(2\mathbf{n}+1)\mathbf{x}}{5\mathbf{n}+}\ldots,$$

are put forward as true theorems, which, however, require more rigorous methods of proof applied to them, especially for the case $m=p/q$.

Example. $$\sqrt[3]{2}=\sqrt[3]{\frac{125+3}{64}}=\frac{5}{4}\sqrt[3]{1+\frac{3}{125}}.$$

Now, $$\sqrt[3]{1+\frac{3}{125}}=1+\frac{1}{125+}\,\frac{1}{1+}\,\frac{1}{\frac{375}{2}+}\,\frac{1}{\frac{4}{3}+}\,\frac{1}{\frac{3125}{4}+}\ldots,$$

and the convergents and limits of error are

$$\left.\begin{matrix}1\\ \\ \end{matrix}\right\},\ \left.\begin{matrix}\frac{126}{125}\\ \\ \end{matrix}\right\},\ \left.\begin{matrix}\frac{127}{126}\\ 4.10^{-7}\end{matrix}\right\},\ \left.\begin{matrix}\frac{47877}{47100}\\ 3.10^{-9}\end{matrix}\right\},\ \left.\begin{matrix}\frac{192143}{190530}\\ 2.10^{-11}\end{matrix}\right\},\ \left.\begin{matrix}\frac{4808937224}{4771070000}\\ 2.10^{-13}\end{matrix}\right\};$$

$$\therefore\ \sqrt[3]{2}=\frac{5}{4}\cdot\frac{4808937224}{4771070000} \quad \text{with error in excess} < 2.10^{-13}$$

$$=\frac{5}{4}.[1\cdot007936841002123] \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

$$=1\cdot2599210512526 \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

$$=1\cdot259921051253 \text{ to twelve places.}$$

* By the use of the calculus for first even convergent.

EXPONENTIAL AND LOGARITHMIC THEOREMS.

Finally, the following illustrations require justification, but yield consistent results :

(i)
$$e^x=\lim_{n\to 0}(1+nx)^{\frac{1}{n}}$$
$$=\lim_{n\to 0}\left\{1+\frac{nx}{n+}\,\frac{(n-1)nx}{2+}\,\frac{(n+1)nx}{3n+}\,\frac{(2n-1)nx}{2+}\cdots\right\}$$
$$=\lim_{n\to 0}\left\{1+\frac{x}{1+}\,\frac{(n-1)x}{2+}\,\frac{(n+1)x}{3+}\,\frac{(2n-1)x}{2+}\cdots\right\}$$
$$=1+\frac{x}{1-}\,\frac{x}{2+}\,\frac{x}{3-}\,\frac{x}{2+}\,\frac{x}{5-}\cdots$$
$$\therefore\ \mathbf{e^x=1+\frac{x}{1+}\,\frac{x}{-2+}\,\frac{x}{-3+}\,\frac{x}{2+}\,\frac{x}{5+}\cdots.}$$

(ii) If $x=1$, $e=1+\left[\frac{1}{1+}\,\frac{1}{-2+}\,\frac{1}{-3+}\,\frac{1}{2+}\,\frac{1}{5+}\,\frac{1}{-2+}\,\frac{1}{-7+}\cdots\right]$,

and the "convergents," for the part in brackets, are

$$1,\ 2,\ \frac{7}{4},\ \frac{12}{7},\ \frac{67}{39},\ \frac{122}{71},\ \frac{921}{536},\ \frac{1720}{1001},\ \frac{16401}{9545},\ \frac{31082}{18089},$$
$$<\ >\ >\ <\ <\ >\ \ >\ \ <\ \ <\ \ >$$

which give a value for $e-1$ lying between

$$1{\cdot}718281822 \text{ and } 1{\cdot}7182818288$$

which is consistent with $e=2{\cdot}7182818285$.

(iii)
$$\log_e(1+x)=\lim_{n\to\infty} n(\sqrt[n]{1+x}-1)$$
$$=\lim_{n\to\infty}\frac{nx}{n+}\,\frac{(n-1)x}{2+}\,\frac{(n+1)x}{3n+}\,\frac{(2n-1)x}{2+}\,\frac{(2n+1)x}{5n+}\cdots$$
$$\therefore\ \mathbf{\log_e(1+x)=\frac{x}{1+}\,\frac{x}{2+}\,\frac{x}{3+}\,\frac{2x}{2+}\,\frac{2x}{5+}\cdot\frac{3x}{2+}\,\frac{3x}{7+}\cdots.}$$

The convergents to this are

$$\frac{x}{1},\ \frac{2x}{2+x},\ \frac{6x+x^2}{6+4x},\ \frac{2(6x+3x^2)}{2(6+6x+x^2)},\ \frac{2(36x+16x^2)}{2(30+36x+9x^2)},$$
$$\frac{2(72x+50x^2+9x^3)}{2(60+90x+36x^2+3x^3)},\ \ldots,$$

where, for instance, $(6x+3x^2)/(6+6x+x^2)$ has an error in defect

$$<\lim_{n\to\infty}\frac{n(n^2-1)(4n^2-1)}{720n^5}x^5<\frac{x^5}{180}.$$

This limit of error apparently is unobtainable from the convergents themselves, but may be proved as follows :

$$(6+6x+x^2)\log_e(1+x)=6x+3x^2+\frac{1\,.\,2}{3\,.\,4\,.\,5}x^5-\frac{2\,.\,3}{4\,.\,5\,.\,6}x^6+\frac{3\,.\,4}{5\,.\,6\,.\,7}x^7-\ldots,$$

where the terms of the series after that containing x^7 are each less than the preceding if $0 < x < 2/3$;

$$\therefore \log_e(1+x) - \frac{6x+3x^2}{6+6x+x^2} < \frac{x^5}{30}\left(1 - \frac{3}{2}x + \frac{12x^2}{7}\right)\Big/(6+6x+x^2)$$

$$< \frac{x^5}{180}, \text{ but positive if } x < 1.$$

Examples.

If $x = \frac{1}{100}$, $\log_{10}\frac{101}{100} = \frac{603\mu^*}{60601}$ with an error in defect $< 3.10^{-13}$,

$\log_{10}101 = 2{\cdot}0043213738.$

If $x = \frac{1}{10}$, $\log_{10}\frac{11}{10} = \frac{63\mu}{661}$ with an error in defect $< 3.10^{-8}$

$= {\cdot}041392666$ " " " " ;

$\therefore \log_{10}11 = 1{\cdot}0413927.$

If $x = \frac{1}{11}$, $\log_{10}\frac{12}{11} = \frac{69\mu}{793}$ with an error in defect $< 3.10^{-8}$

$= {\cdot}037788549$ " " " " ;

$\therefore \log_{10}12 = 1{\cdot}079181215$ " " " $< 6.10^{-8}$

$= 1{\cdot}0791813.$

These results are consistent with those given in 7-figure tables.

Theoretically it remains to be shown rigorously that these theorems are true or that the limits of truth should be determined. One point arises at once: the C.F. for $\sqrt[n]{1+x}$ expansion is convergent apparently so long as $x < 4$; does then the C.F. represent $\sqrt[n]{1+x}$ and not the expanded form between $x = 1$ and $x = 4$?

NOTE.—Since writing the above I have succeeded in obtaining a rigorous proof, except for a small detail of convergency, that the continued fraction is equivalent to the Binomial series; the proof is similar to the method of expressing the quotient to two hypergeometrics as a C.F., and was suggested by it.

(*To be continued.*)

MATHEMATICAL NOTES.

473. [V. 1. a. δ.] In an Accuracy Test-paper recently set to a number of Public Schools, the following questions were given:

Divide 2·7 by 514·3 to five decimal places (0·00525).

Divide 1·63 by 8·54 to three " (0·191).

The answers, given on the paper, are correct to the fifth and third places respectively (actual division giving 0·005249... and 0·1908...).

An official pronouncement was made that the correct answers to the questions are 0·00524 and 0·190.

I would be grateful to know if this view is supported by the majority of mathematicians. A NON-MATHEMATICAL MEMBER.

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* Taking $\mu = {\cdot}4342944819$ from Chambers. For closer approximations μ would have to be calculated from the convergents above or closer ones.

474. [V. 1. a. δ.] In the same paper (*v.* Note 473) another question asks for the value of $(23 \cdot 7 \times 0 \cdot 315)^2$ to three significant figures.

May I enquire how a boy is to determine the degree of accuracy in such a question, unless he assumes that the given numbers are correct to three significant figures, in which case the answer cannot be relied upon as being correct to three figures? PSEUDO-ACCURACY.

475. [V. 1. a. δ.] *The Use of Brackets in Arithmetic.*

The Committee on the Teaching of Arithmetic in Public Schools say in their last report (*Gazette*, viii. 238):

> "The only convention that should be required is that which governs the interpretation of such expressions as $10\frac{1}{2} - 2\frac{1}{4} \times 1\frac{1}{3} + 3\frac{5}{8}$. This same convention is used in Algebra; *e.g.* in $a - bc + d$."

The second sentence is obviously incorrect: the corresponding expression in algebra would not be $a - bc + d$, but

$$a - b \times c + d.$$

The argument of the Committee would have been more logical if they had said:

> "In algebra we are able to dispense with brackets in some cases by dispensing with the sign of multiplication; thus, instead of $a - (b \times c) + d$ we can write $a - bc + d$. We cannot do this in arithmetic—we cannot, for instance, replace $7 - (2 \times 3) + 1$ by $7 - 23 + 1$—and therefore we must use brackets in all cases of ambiguity."

What I have never been able to understand is why there should be all this fuss about introducing a couple of brackets. They are not troublesome to write or print, and they avoid the necessity of learning by rote the arbitrary and (to the young student) apparently meaningless rule that "multiplications and divisions are to be performed before additions and subtractions." I write with some feeling; for, as a boy, I never could remember the rule, and I always had to look it up before going in for an arithmetic examination.

Mr. C. S. Jackson and Mr. A. Lodge have tried (*ibid.* 246-8) to defend the rule. They both seem to think that any objection that applies to $a - b \times c + d$ applies also to $a - bc + d$. Mr. Jackson seems to suggest (middle of p. 247) that the young student either must learn that $17 - 3 \times 4$ means $17 - (3 \times 4)$ or must be led to suppose that it is capable of two interpretations: he omits the third possibility, that it should never be used at all. Mr. Lodge says that the interpretation of $a \times b + c \times d + e \times f$ as meaning $(a \times b) + (c \times d) + (e \times f)$ rather than $a \times (b + c) \times (d + e) \times f$ is "absolutely fundamental." In what way "fundamental"? It is true, probably, that most of our calculations result in the addition or subtraction of terms that are obtained by multiplication or division, rather than the other way round; but this is a matter of experience, which the young pupil has not had. I cannot see in what way the fact (if it is a fact) can be regarded as forming part of the foundation of algebraical reasoning.

If some particular boy finds a difficulty in understanding that $ab - c$ does not mean $a(b - c)$, why not let him write it $(ab) - c$?

It is not, as Mr. Lodge seems to think, a question of "carelessness." It is, so far as the pupil is concerned, a question of overloading the memory by burdening it with a useless rule. If a boy's memory must be exercised, why not teach him something more useful, such as "She went into the garden to cut a cabbage to make an apple-pie"? And, so far as the teacher is concerned, it is a question of discriminating between the unessential and the essential. W. F. SHEPPARD.

476. [K¹. 2. a, b.] 466, *Math. Gazette*, *March* 1916, p. 251. The theorem is given in GALLATLY'S *Modern Geometry of the Triangle*, 2nd ed., 1913, § 55, as due to A. C. Dixon.

The SIMSON lines of the ends of the diameter through the in-centre intersect at right angles in the FEUERBACH point. This theorem is fundamental in the geometrical interpretation of the Cubic Transformation, where all the properties of the triangle correspond to an Elliptic Function relation in the *Fundamenta Nova*. So that the Geometry of the Triangle does not deserve the contempt expressed for it in some quarters as mere trifling, leading nowhere. G. GREENHILL.

477. [R. 5. a.] *Attraction by Spheroids.*

In connection with this subject I have recently devised the following formulae, which appear to have some advantages in compactness over those usually given.

In what follows, in every case, $\sin\beta=e$, the eccentricity, $K=\pi\rho G$, where ρ is the density and G the constant of gravitation, y and x are the coordinates of the attracted particle, and Y and X are the components of the attraction of the spheroid parallel to the y and x-axes, the y-axis being always taken to be that of the figure of revolution.

OBLATE SPHEROID.

For a particle on the surface,

$$Y=y\,4\cos\beta\operatorname{cosec}^3\beta[\tan\beta-\beta]K, \quad\text{.........................(1)}$$

$$X=x\,2\cos\beta\operatorname{cosec}^3\beta[\beta-\sin\beta\cos\beta]K. \quad\text{.................(2)}$$

For an external particle,

$$Y=y\,4\cos\beta\operatorname{cosec}^3\beta[\tan\gamma-\gamma]K, \quad\text{.........................(3)}$$

$$X=x\,2\cos\beta\operatorname{cosec}^3\beta[\gamma-\sin\gamma\cos\gamma]K. \quad\text{..................(4)}$$

In (1), (2), (3), (4), a is the semi-major axis along the x-axis; b the semi-minor axis along the y-axis, so that $\sin\beta=\sqrt{(a^2-b^2)}/a$.

In (3) and (4) the positive root of

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}=1 \text{ gives } u,$$

and then $$\sin\gamma=\frac{a\sin\beta}{\sqrt{(a^2+u)}} \text{ gives } \gamma.$$

It will be seen that (3) and (4) differ from (1) and (2) only inside the square brackets, γ being substituted for β.

PROLATE SPHEROID.

For a particle on the surface,

$$Y=y\,4\cos^2\beta\operatorname{cosec}^3\beta[gd^{-1}\beta-\sin\beta]K, \quad\text{......................(5)}$$

$$X=x\,2\cos^2\beta\operatorname{cosec}^3\beta[\tan\beta\sec\beta-gd^{-1}\beta]K. \quad\text{............(6)}$$

For an external particle,

$$Y=y\,4\cos^2\beta\operatorname{cosec}^3\beta[gd^{-1}\gamma-\sin\gamma]K, \quad\text{......................(7)}$$

$$X=x\,2\cos^2\beta\operatorname{cosec}^3\beta[\tan\gamma\sec\gamma-gd^{-1}\gamma]K. \quad\text{............(8)}$$

In (5), (6), (7), (8), a is the semi-minor axis along the x-axis; b is the semi-major axis along the y-axis, so that

$$\sin\beta=\sqrt{(b^2-a^2)}/b.$$

In (7) and (8), we have u from the positive root of

$$1=\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u},$$

and then γ from
$$\sin\gamma=\frac{b\sin\beta}{\sqrt{(b^2+u)}}.$$

It will be noticed that (7) and (8) differ from (5) and (6) only inside the square brackets, γ being substituted for β.

To illustrate the brevity of the formulae, I may point out that (8) uses only eight literal symbols, while in a recent text-book the corresponding formula is given as

$$X=\frac{x\,2\pi\rho G}{e^3}\left[\frac{ae}{\sqrt{(a^2+u)}}\sqrt{\left(1-e^2+\frac{a^2e^2}{a^2+u}\right)}\right.$$
$$\left.-(1-e^2)\log\left(\frac{ae}{\sqrt{(1-e^2)}\sqrt{(a^2+u)}}\right)+\sqrt{\left(1+\frac{a^2e^2}{(1-e^2)(a^2+u)}\right)}\right],$$

in which thirty-one symbols appear. Of course u has to be found before we can solve this. C. T. WHITMELL.

Invermay, Hyde Park, Leeds.

478. [K. 21. b.] (Vol. vii. p. 108, No 393.) A shorter proof is as follows:

Since DB bisects EC at right angles, the triangle ECB is isosceles and similar to BCO.

$$\therefore\ C\hat{O}B=C\hat{B}E=2\,.\,D\hat{B}E=2\,.\,D\hat{O}A.$$

W. J. DOBBS.

479. [V. 1.] *Extracted from a Review* (*unpublished*):

"There is much that he does not say that he means, that he knows you know he means, and so you cannot contradict what he does not say, what you know he means to say—and yet you cannot agree with what he does say, for you know that that will be taken to mean that you agree with what he does not say as well, and to the latter you are firmly opposed. I hope the reader will survive this. That is how I felt after reading 'X.'"

EDITOR.

480. [L². 2. c.] (Vol. viii. p. 50.) The following remarks are suggested by Note 443:

It is or ought to be well known that the result of eliminating x from

$$\Sigma lx=0 \quad\text{.....................................(1)}$$

and
$$\Sigma ax^2+2\Sigma fyz=0, \quad\text{..............................(2)}$$

is
$$C'y^2+B'z^2-2F'yz=0,$$

where A' etc., F' etc. are the co-factors of a etc., f etc. in the bordered discriminant

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & \end{vmatrix},\ =\Delta,\text{ say}$$

of (1), (2), for this result was given more than sixty years ago by Sylvester.

Hence if x_1, y_1, z_1; x_2, y_2, z_2 are the two solutions of (1), (2), we have

$$\frac{x_1x_2}{A'}=\frac{y_1y_2}{B'}=\frac{z_1z_2}{C'}=\frac{y_1z_2+y_2z_1}{2F'}=\frac{z_1x_2+z_2x_1}{2G'}=\frac{x_1y_2+x_2y_1}{2H'}.$$

Also, by a theorem of Jacobi, we have

$$B'C' - F'^2 = -l^2\Delta.$$

Hence if θ be the angle between the lines given by (1), (2), we have

$$\tan\theta = \frac{2\Sigma l^2\sqrt{\Delta}}{\Sigma A'}.$$

Thus the lines are real or imaginary as $\Delta > 0$ or $\Delta < 0$, whilst the condition for perpendicularity, $\Sigma A' = 0$, is clearly

$$\Sigma al^2 + 2\Sigma fmn = \Sigma a \Sigma l^2.$$

The same piece of algebra of course occurs in finding the condition that a line may touch a conic or be cut harmonically by two conics. E. J. NANSON.

481. [E. 5.] *The Integrals* $\int_0^{\frac{\pi}{2}} \frac{\sin^2 mx}{\sin^2 x}\,dx$, $\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx$, $\int_0^{\infty} \frac{\sin x}{x}\,dx$.

These integrals (m being an integer in the first) are of importance in connection with the theory of Fourier series. The following very elementary method of finding their values is suggested by the method applied by Prof. A. C. Dixon* to the third integral.

1. Denoting the first integral by S_m, an integration by parts gives

$$S_m = m\int_0^{\frac{\pi}{2}} \sin 2mx \cot x \,dx.$$

Hence $$S_m/m - S_{m-1}/(m-1) = 2\int_0^{\frac{\pi}{2}} \cos(2m-1)x \cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \{\cos(2m-2)x + \cos 2mx\}\,dx = 0,$$

and therefore $S_m/m = S_1 = \pi/2$.

2. Since $\sin^2 x/x^2$ is positive, and

$$\int_l^{\infty} \sin^2 x\,dx/x^2 < \int_l^{\infty} dx/x^2 = 1/l,$$

$\int_0^{\infty} \sin^2 x\,dx/x^2$ exists, and is the limit of $\int_0^l$ when l tends to infinity in any manner. We can therefore take $l = m\pi/2$, m being an integer.

Now $$\int_0^{\frac{m\pi}{2}} \frac{\sin^2 x}{x^2}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin^2 mx}{mx^2}\,dx.$$

Since $\operatorname{cosec}^2 x > x^{-2} > \cot^2 x$ in the interval $(0, \pi/2)$, the value of the integral last written is intermediate between $S_m/m = \pi/2$, and

$$\frac{1}{m}\int_0^{\frac{\pi}{2}} \sin^2 mx \cot^2 x\,dx = \frac{1}{m}\left(S_m - \int_0^{\frac{\pi}{2}} \sin^2 mx\,dx\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4m}.$$

The required limit is therefore $\pi/2$.

3. Prof. H. C. M'Weeney has pointed out to me that the preceding argument, followed by an integration by parts, gives $\int_0^{\infty} \sin x\,dx/x$ very simply.

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* *Math. Gazette*, Note 367, 1912 p. 223.

In fact,
$$\int_a^b \frac{\sin^2 x}{x^2}\,dx = -\left[\frac{\sin^2 x}{x}\right]_a^b + \int_a^b \frac{\sin 2x}{x}\,dx$$
$$= -\left[\frac{\sin^2 x}{x}\right]_a^b + \int_{2a}^{2b} \frac{\sin x}{x}\,dx.$$

Allowing a and b to tend to zero and infinity respectively, we get the result. M. F. Egan.

Dublin.

482. [H. 5. a.] *Linear Differential Equations with Constant Coefficients.*

The method usually employed to obtain the solutions of the differential equation $\frac{d^2y}{dx^2}+a\frac{dy}{dx}+b=0$, in the case where the constants a and b are such that the two roots of the quadratic $m^2+am+b=0$ coincide, involves some very unconvincing arguments about "infinite constants."

To avoid these we may proceed as follows:

Denote the operator $\left(\frac{d^2}{dx^2}+a\frac{d}{dx}+b\right)$ by D.

Then
$$De^{mx}=(m^2+am+b)e^{mx}$$
$$=(m-\alpha)^2e^{mx},$$

if α is the repeated root of the quadratic, and

$$D\left(\frac{\partial}{\partial m}e^{mx}\right)=\frac{\partial}{\partial m}(De^{mx})=2(m-\alpha)e^{mx}+(m-\alpha)^2xe^{mx}.$$

Thus De^{mx} and $D\left(\frac{\partial}{\partial m}e^{mx}\right)$ both vanish if $m=\alpha$.

But $\frac{\partial}{\partial m}e^{mx}=xe^{mx}$.

Therefore $e^{\alpha x}$ and $xe^{\alpha x}$ are the required solutions of $Dy=0$.

Equations of higher order may be treated in a similar manner.

If the equation in m has three coincident roots, we use $\frac{\partial^2}{\partial m^2}e^{mx}$, and so on in general. H. Piaggio.

483. [D. 6. b. d.] *Note on Napier's Logarithms.*

In the many papers which have appeared in connection with the Napier Tercentenary of 1914, it has been frequently pointed out that in Napier's system of logarithms the logarithm of unity is not zero. I have not noticed, however, anywhere a statement of the actual value of Log 1 in Napier's system, although this is readily found from the well-known relation between Napier's logarithms and Hyperbolic logarithms. The following may therefore be of interest.

The fundamental property of Napier's logarithmic function, $f(x)$, is that if $a:b=c:d$, then $f(a)-f(b)=f(c)-f(d)$, or, in particular, putting $a=x$ $b=1$, $c=xy$, $d=y$,
$$f(x)+f(y)=f(xy)+f(1).$$

Differentiating partially with regard to x and y separately, we have
$$f'(x)=yf'(xy) \quad \text{and} \quad f'(y)=xf'(xy),$$
whence $xf'(x)=yf'(y)=$ a constant $=a$; and, on integrating, we have
$$f(x)=a\log_e x+b.$$

We shall denote Napier's logarithms by Log, and logarithms to base e by log.

Now, even supposing that we know nothing about the mode of construction of Napier's system, but have only his tables, we can determine the constants a and b if we know only the Logs of two distinct numbers.

We find from the tables

$$\text{Log}\,10^7=0 \quad\text{and}\quad \text{Log}\,9995000=5001{\cdot}25.$$

Hence we have $$7a\log 10+b=0,$$

and $$a\log 9995000+b=5001{\cdot}25.$$

Therefore $$a(7\log 10-\log 9995000)=-5001{\cdot}25.$$

The coefficient of $a=-\log{\cdot}9995={\cdot}000500125$, as is easily calculated from the logarithmic series.

Hence $a=-10^7$, *exactly*, so far as the figures go.

Then $b=-7a\log 10=7\times 10^7\times 2{\cdot}302585093=161180956{\cdot}5$.

The relationship between Napier's logarithms and Hyperbolic logarithms is therefore

$$\begin{aligned}\text{Log}\,x&=10^7(\log 10^7-\log x)\\&=161\,180\,957-10^7\log x.\end{aligned}$$

Then $$\text{Log}\,1=10^7\log 10^7=161\,180\,956{\cdot}5.$$

This may be verified by taking examples from the tables.*

Thus $\text{Log}\,2909=81\,425\,681$ and $\text{Log}\,2909^2=\text{Log}\,8462281=1\,669\,663$;

therefore $$\text{Log}\,1=2\,\text{Log}\,2909-\text{Log}\,2909^2=161\,181\,699.$$

Again, by interpolating,
$$\begin{aligned}\text{Log}\,11636&=\ \ 67\,562\,746,\\ \text{Log}\,100&=115\,129\,512,\\ \text{Log}\,1163600&=\ \ 21\,510\,657\,;\end{aligned}$$

therefore
$$\begin{aligned}\text{Log}\,1&=\text{Log}\,11636+\text{Log}\,100-\text{Log}\,1163600\\&=161\,181\,601.\end{aligned}$$

D. M. Y. SOMMERVILLE.

484. [E. 5; V. a. λ.] *Further remarks on the integral* $\int_0^\infty \frac{\sin x}{x}\,dx$.

I have been asked for my opinion of the difficulty of Prof. Dixon's proof, as compared with those of the proofs I discussed in my previous note on this subject (*Gazette*, v. p. 98). The conciseness and ingenuity of the proof no one could question. Dr. Bromwich and Mr. Whipple have also sent me proofs that I had not seen or forgot to mention, and it seems appropriate that I should attempt to assign marks, according to the principles I then stated, to all three. Before I do so, however, I should like to mention two general principles suggested by Dr. Bromwich as comments on my system of marking, as I am myself of opinion that their acceptance makes my method fairer and easier to apply.

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*I have no access to a copy of Napier's tables, but on calculating the values of Napier's logarithms from the corresponding hyperbolic logarithms, it appears that these numbers should be corrected as follows:

$$\begin{aligned}\text{Log}\,2909&=81\,425\,310 &\qquad \text{Log}\,11636&=\ \ 67\,562\,366\\ \text{Log}\,2909^2&=\ \ 1\,669\,663 &\qquad \text{Log}\,100&=115\,129\,255\\ && \text{Log}\,1163600&=\ \ 21\,510\,664\end{aligned}$$

The hyperbolic logarithms of 2, 3 and 10 which are used here I have obtained from Adams' paper in *Proc. R. Soc.* (1878) **27**, p. 92. The value of log 2909 I have calculated from the formula $\log 2909=\log 3000+\log\left(1-\frac{91}{3000}\right)$. The values are

$$\begin{aligned}\log 2&=\ {\cdot}69314\,71806 &\qquad \log 10&=2{\cdot}30258\,50930\\ \log 3&=1{\cdot}09861\,22887 &\qquad \log 2909&=7{\cdot}97556\,46585\end{aligned}$$

The first is, that when a proof involves two inversions resembling one another both in principle and detail, then the mark for the second should be reduced by one-half, *e.g.* from 10 to 5.

The second rests on the following considerations.

Let us consider, for clearness, the case of an integration of a series over a finite range. Then certain integrations stand out by themselves as peculiarly easy to justify, viz. those in which the series is uniformly *and absolutely* convergent in virtue of "Weierstrass's M-Test" (Bromwich, *Infinite Series*, p. 113 and also p. 207). Such a case is that involved in

$$\int_0^{\pi} \frac{\cos n\theta\, d\theta}{1-2p\cos\theta+p^2} = \frac{1}{1-p^2}\int_0^{\pi}\left\{1+2\sum_1^{\infty} p^k \cos k\theta\right\}\cos n\theta\, d\theta = \frac{\pi p^n}{1-p^2}$$

(where, of course, $|p|<1$). Cases in which the series is uniformly but not absolutely convergent are intrinsically more difficult, depending as they do on one or other of the tests called by Dr. Bromwich "Abel's" and "Dirichlet's" (*l.c.* pp. 113-4), and, in the long run, on "Abel's Lemma" (p. 54). An example of this kind is the integration in what I called in my former note, "Mr. Berry's second proof."

A similar distinction is to be made in regard to proofs of the continuity of an infinite series, and, *mutatis mutandis*, proofs of the legitimacy of other types of inversions, although it cannot, of course, always be presented in so clear-cut a form.

Dr. Bromwich suggests that 15 marks at least should be assigned for any inversion of the more difficult type, and I am disposed to adopt this suggestion. The 10 or 15 marks are, of course, only intended as a rough standard by which to make a beginning, and would generally be modified in any particular case.

On looking through my former note, in the light of these remarks, I am disposed to make the following alterations in my original marks.

Proof 2.* Assign 25 instead of 20 for the inversion of the integrations, making the total 37 instead of 32. Also add 5 to the total for Proof 5, making it 45.

Proof 4. Add 5 marks in respect of the first inversion, making the total 45.

Proof 7. Deduct say 28, leaving 80.

Dr. Bromwich's proof, mentioned in the last footnote of my former note (which we may call Proof 8), I should now mark at about 50 instead of 45. In so far as it is concerned with this particular integral, it differs little from Proof 4. I proceed to consider the three new proofs.

Proof 9 (*Dr. Bromwich's second proof*). We have

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} e^{-x\sin\theta}\cos(x\cos\theta)\,d\theta &= \int_0^{\frac{1}{2}\pi} R\left[e^{ixe^{i\theta}}\right]d\theta\\ &= \int_0^{\frac{1}{2}\pi}\left(1-x\sin\theta-\frac{x^2}{2!}\cos 2\theta+\frac{x^3}{3!}\sin 3\theta+\dots\right)d\theta\\ &= \frac{1}{2}\pi - x + \frac{x^3}{3.3!} - \frac{x^5}{5.5!} + \dots\\ &= \frac{1}{2}\pi - \int_0^x \frac{\sin t}{t}\,dt.\end{aligned}$$

But the original integral is numerically less than

$$\int_0^{\frac{1}{2}\pi} e^{-2x\theta/\pi}\,d\theta = \frac{\pi}{2x}\left(1-e^{-x}\right) < \frac{\pi}{2x},$$

* The numbers are those assigned to the proofs in my final list (*Gazette*, v. p. 98).

and the result follows on making n tend to ∞. This proof is due to Schlömilch, *Übungsbuch der höheren Analysis*, p. 173. It is to be observed that it and Proofs 2 and 5 are really variations of the same theme, and closely connected with the proof by contour integration. This being so, it is clear that the straightforward proof by contour integration should have a better mark than either; and I deduct 2 from the marks for Proofs 3 and 6, which I had perhaps marked a little severely.

As both integrations of a series in Proof 9 are of the simplest type, I give 10 marks for the first integration, 5 for the second, 5 for the summation of the trigonometrical series and 10 for the final step in the argument, making 30 in all. I add 5 more because the proof has a certain artificiality, though less than Mr. Michell's, as contrasted with such proofs as 1 and 4; the total is therefore 35.

Proof 10 (*Mr. Whipple's proof*). This depends on the transformations

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{h\to 0} h\sum_1^\infty \frac{\sin nh}{nh} = \lim_{h\to 0} \frac{1}{2}(\pi - h) = \frac{1}{2}\pi.$$

This proof has the great merit of being absolutely natural and straightforward in idea; unfortunately, however, the inversion is one of a very difficult character.* Also the assumption of a knowledge of the sum of the trigonometrical series is a rather serious matter. I cannot give less than $30+15=45$ marks. At the same time, the proof seems to me a most interesting and suggestive one.

Proof 11 (*Prof. Dixon's proof*).† This proof I find harder to mark than any. It is exceedingly concise, and it avoids all inversions of whatever kind in the most ingenious way; its difficulties are, indeed, of a kind which rather baffle my rules. Rules or no rules, I cannot regard it as on a par with Proof 1, nor, I think, as being really quite as simple as 3 or 9. At a rough guess, let us say 36. As it is absurd to try to make very fine distinctions, I alter the marks of 9 and 2 from 35 and 37 to 36 each also. My final list, then, works out as follows:

1.	Proof 1	(*Mr. Berry's second*)	28.
2.	Proof 3	(*Mr. Berry's form of his third*)	32.
3.	Proof 9	(*Dr. Bromwich's second*)	36.
	Proof 11	(*Prof. Dixon's*)	36.
	Proof 2	(*My form of Mr. Michell's*)	36.
6.	Proof 6	(*Ordinary form of Mr. Berry's third*)	40.
7.	Proof 4	(*Mr. Berry's first*)	45.
	Proof 5	(*Mr. Michell's original*)	45.
	Proof 10	(*Mr. Whipple's*)	45.
10.	Proof 8	(*Dr. Bromwich's first*)	50.
11.	Proof 7	(*Prof. Nansen's*)	80.

As I remarked before, Proofs 7 and 8, especially the first, are heavily penalised because they include proofs of a great deal of extraneous matter. As regards absence of serious theoretical difficulty, I think 9 is possibly the simplest of all. I regret the low position of 4 and 10; each is a good proof, proceeding to its goal by a straight and natural route, but unfortunately meeting with difficult theoretical obstacles. Finally, I may point out that Proof 1 is sometimes presented in a somewhat different form (see Bromwich, *Infinite Series*, p. 468), in which the integration of a series may be made to depend upon an "*M*-test," and which may possibly deserve a slightly lower mark.

G. H. Hardy.

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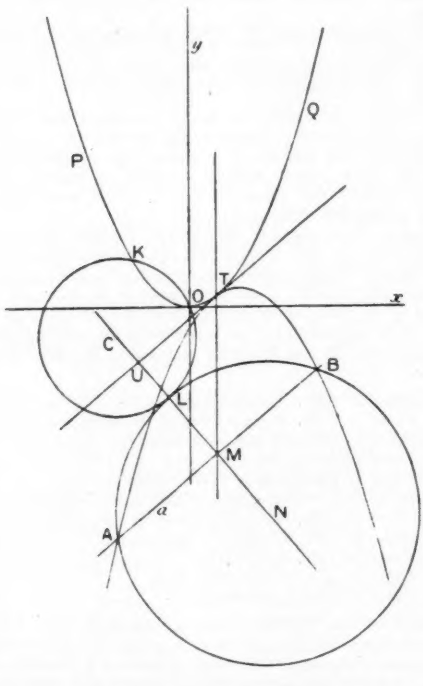
*See a paper by Dr. Bromwich and myself in the *Quarterly Journal*, vol. xxxix. p. 222; also Cesàro, *Algebraische Analysis*, p. 699.

† *Gazette*, vi. p. 225.

485. [X. 4. b. β.] *The Graphical Solution of a Cubic Equation with Complex Roots.*

The following is an application to the solution of cubic equations with imaginary roots, of methods already given of the solution of quadratic equations, in Hamilton and Kettle's *Graphs and Imaginaries.* It is there shown that:

(I) If POQ be any parabola (see figure) and a (or AB) is any straight line meeting POQ in imaginary points, the coordinates of the intersections of parabola and line are $(x \pm i\xi, y \pm i\eta)$, if the "shadow parabola" (see below) of the line meets that line in $(A, B) \equiv (x \pm \xi, y \pm \eta)$;



(II) That a meets any circle KOL in the points whose coordinates are $(x \pm i\xi, y \pm i\eta)$ if the "shadow circle" of a meets a in $(x \pm \xi, y \pm \eta)$.

The "shadow parabola" of POQ corresponding to a is an equal parabola touching POQ at T, where the tangent is $\parallel AB$. The "shadow" is upside down as in figure.

The "shadow circle" of KOL corresponding to a has N as centre and touches KOL, where $CN \perp^{r} AB$ and $MN = CL$.

Now the roots of the equation

$$x^3 + px + q = 0 \qquad \text{.......................................(1)}$$

are the abscissae, other than the origin, of the intersections of the parabola

$$y=x^2 \quad\text{..................................(2)}$$

and the circle $$x^2+y^2+(p-1)y+qx=0. \quad\text{..............................(3)}$$

(See Hardy, *Pure Mathematics*, Examples on Chap. II.)

Let POQ be the parabola (2); and let KOL be the circle (3).

The real root of (1) is the abscissa of K.

Supposing (1) to have imaginary roots, the join of the imaginary intersections of (2) and (3) is a real line parallel to that tangent (TU) to POQ, which makes the same angle with the axis of POQ as KO does.

Draw $CU \perp^r TU$, and produce to M, where the diameter conjugate to TU meets it.

Mark off $MN=CL$, and draw the circle of centre N and radius NL.

Draw AMB through M and parallel to the tangent to POQ at T, meeting the circle centre N in A, B.

Then if A, B are the points $(x\pm\xi, y\pm\eta)$, the imaginary intersections of (2) and (3) are $(x\pm i\xi, y\pm i\eta)$. Hence the imaginary roots of (1) are $x\pm i\xi$.

For, the "shadow parabola" corresponding to the point T will clearly pass through the points A, B determined as above.

Then by the principle (I) (see above), the points A, B determine the intersections of the line a with the parabola POQ. Also by (II) the same points A, B determine the intersections of a with the circle KOL.

Hence A, B determine the intersections of the parabola POQ with the circle KOL, which are what we require.

E. R. HAMILTON.

486. [I. 5.] *The Definition of a Complex Number.*

In the March number of the *Gazette* Dr. Hardy discusses the introduction of complex numbers by means of an operator O, such that $O\{O(a)\}=-a$. He says that this method has been adopted in a recently published book, and develops in detail the objection that $O\{O(a)\}=-a$ does not define any unique operation.

I know of only one recently published book, Dr. Bowley's *Pure Mathematics*, which adopts this method of development, and as it has been reviewed by Dr. Hardy in the *Cambridge Review*, it is doubtless the one to which he refers. But in that book the operator O is not applied to a complex number, but only to real or purely imaginary numbers, so that Dr. Hardy's analysis does not apply.

Of course the development of imaginary and complex numbers in an elementary text-book is, as far as my experience goes, bound to be open to criticism. So is the development of negative numbers, at the stage at which they are generally introduced, unless an unusual amount of space is devoted to it. In the book in question the whole of the mathematics, beyond matriculation standard, which the author considers necessary for the non-specialist is given in 269 pages, so that space must be an important consideration.

In spite of some objections, I have found the "operator" method a useful one for young students, as it helps them to realise that there is a difficulty. They are apt to regard ι as "the square root of -1" and to settle the whole matter in one line. And it is open to doubt whether the idea that $a+\iota b$ is merely a symbolic equivalent for a pair of real numbers (a, b) gives the best way of introducing complex numbers to young students.

G. W. PALMER.

487. [L[1]. 3. d.] *Parabolic Asymptotes.*

To find the parabolic asymptote of

$$(y-2x)^2(y^2-x^2)+y^3+2x^2y+y^2+x^2+ax+by+c=0.$$

Write the equation

$$(y-2x)^2+\frac{y^3+2x^2y}{y^2-x^2}+\frac{y^2+x^2}{y^2-x^2}+\frac{ax+by+c}{y^2-x^2}=0.$$

Put $y=2x+b$ in the second term, $y=2x$ in the third, and neglect powers of b above the first.

We get

$$(y-2x)^2+\frac{12x^3+14x^2b+\text{higher powers of } b}{3x^2+4xb+b^2}+\frac{5x^2}{3x^2}+0=0.$$

Divide denominator into numerator in second term.

$$\begin{array}{l} 3x^2+4xb+\ldots\ )\ 12x^3+14x^2b+\ldots\ (\ 4x-\frac{2}{3}b \\ \qquad\qquad\qquad\ \ 12x^3+16x^2b+ \\ \qquad\qquad\qquad\qquad\quad -\ 2x^2b+\text{etc.} \\ \qquad\qquad\qquad\qquad\quad -\ 2x^2b+\text{etc.} \end{array}$$

and the equation of the asymptote becomes

$$(y-2x)^2+4x-\tfrac{2}{3}b+\tfrac{5}{3}=0.$$

Replacing b by $y-2x$, we get

$$(y-2x)^2+4x-\tfrac{2}{3}(y-2x)+\tfrac{5}{3}=0,$$

$$(y-2x)^2-\tfrac{2}{3}y+\tfrac{16}{3}x+\tfrac{5}{3}=0,$$

or

$$3(y-2x)^2-2y+16x+5=0.$$

G. H. Bryan.

REVIEWS.

A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytical Functions; with an Account of the Principal Transcendental Functions. Second Edition, completely revised. By E. T. Whittaker and G. N. Watson. Pp. viii + 560. Cloth, 18s. net. 1915. (Cambridge: University Press.)

In the first edition of this work, which was written by Prof. Whittaker alone and published in 1902, it was stated that the first half was devoted to "those methods and processes of higher mathematical analysis which seem to be of greatest importance at the present time," while in the second half "the methods of the earlier part are applied in order to furnish the theory of the principal functions of analysis." This description holds for this second edition. The original preface is not given in the new edition, and the book has grown by nearly 200 pages. Mr. Watson has added entirely alone the fourth, eleventh, and thirteenth chapters, and has worked with Prof. Whittaker at many of the others. Thus the tenth, nineteenth, and twenty-first chapters are presumably added by the labour of both Prof. Whittaker and Mr. Watson, and the eighth chapter is a new one, but there was a chapter in the first edition on "Asymptotic Expansions." Besides this, the first few chapters are greatly altered from the first edition. On the whole, the alterations are for the better, as the important questions lying at the bottom of Goursat's proof of Cauchy's theorem, about which the first edition was notoriously defective, are here treated with precision. With regard to the rest of the changes in the first few chapters, which are chiefly prompted by advances in modern logic, it cannot be said that they are by any means satisfactory either to a learner or to a teacher. But this does not affect the more important—at least in the authors' eyes—part of the book, and can be, moreover, easily rectified by a study of such a book as Mr. Hardy's *Course*. The following is an account of the contents of the work: Part I. The Processes of Analysis. Chapter I. Complex Numbers; II. The Theory of Convergence; III. Continuous Functions and Uniform Convergence; IV. The Theory of Riemann Integration;

V. The Fundamental Properties of Analytic Functions ; Taylor's, Laurent's, and Liouville's Theorems ; VI. The Theory of Residues ; Application to the Evaluation of Definite Integrals ; VII. The Expansion of Functions in Infinite Series ; VIII. Asymptotic Expansions and Summable Series ; IX. Fourier Series ; X. Linear Differential Equations ; XI. Integral Equations. Part II. The Transcendental Functions. XII. The Gamma Function ; XIII. The Zeta Function of Riemann ; XIV. The Hypergeometric Function ; XV. Legendre Functions ; XVI. The Confluent Hypergeometric Function ; XVII. Bessel Functions ; XVIII. The Equations of Mathematical Physics ; XIX. Mathieu Functions ; XX. Elliptic Functions ; General Theorems and the Weierstrassian Functions ; XXI. The Theta Functions ; XXII. The Jacobian Elliptic Functions.

I should like to make two suggestions. The first is that there should be a treatment of Cauchy's theory of functions based on the consideration of circles and straight lines alone as contours : the exact treatment of contours in general is so puzzling to a student. It would be quite possible to explain contour integration simply and rigidly for contours composed of a finite number of straight lines and arcs of circles, and then to show by examples that there are difficulties in the general notion of curve, but that the means just mentioned are quite sufficient for all the cases usually met. The book under review is very much simpler in this respect than Mr. Watson's Cambridge Tract on complex integration, but it is so only because much that is logically necessary is omitted. In the second place, it seems that the generalisations of functions are not given in a style that is very illuminating to a student (see, *e.g.*, pp. 235, 238, 349-351). The non-heuristic method, to take a case which is a simple analogy of those dealt with here, of suddenly producing a real or complex integral and verifying that it is a generalisation of the function n is not very educative. One might tend to regard this method as a characteristic of Prof. Whittaker's teaching alone if it were not for a similar method used by Mr. Watson (p. 260). But still this work has shown itself in its first edition to be very stimulating. It certainly deserves to rank high among Cambridge text-books.

The General Theory of Dirichlet's Series. By G. H. HARDY and MARCEL RIESZ. No. 18 of the Cambridge Tracts in Mathematics and Mathematical Physics. Pp. viii + 78. 3s. 6d. net. 1915. (Cambridge: University Press.)

Dirichlet's series—or rather a special kind of them—were first introduced into analysis by Dirichlet, primarily with a view to applications in the theory of numbers. Somewhat generalised, they may be defined as infinite series of terms $a_n \exp(-\lambda_n s)$, where the λ's are real increasing numbers and s is a complex variable. The first theorems involving complex values of s were due to Jensen in 1884 and 1888 ; and the first attempt to construct a systematic theory of functions defined by Dirichlet's series was made by Cahen in 1894 in a manner which, in spite of or possibly because of errors, stimulated most of the later researches in the subject. The theory of the functions in question presents very curious differences from the theory of analytic functions, which may, of course, be studied from the point of view of being defined by series of powers. In the first place, the region of convergence of a Dirichlet's series is a semi-infinite plane, and in the second place, a Dirichlet's series convergent in a portion of the plane only may represent a function regular in a wider region of it or even all over it, and the result is that many of the peculiar difficulties which attend the study of power series on the circle of convergence are extended, in the case of Dirichlet's series, to wide regions of the plane or even the whole of it (pp. 9-10). After giving the elementary theory of convergence, the authors deal with the formula for the sum of the coefficients of a Dirichlet's series ; the summation of divergent Dirichlet's series by "typical means" (of which this is the first systematic account, and which is chiefly due to Dr. Riesz) ; theorems on the summability of these series ; further developments of the theory of functions represented by them ; and the multiplication of Dirichlet's series (which has specially important applications in the analytical theory of numbers). There is, finally, a very

full bibliography which supplements that given up to 1909 in Landan's classical *Handbuch der Lehre von der Verteilung der Primzahlen.*

The work of Mr. Hardy, among others, has brought, of recent years, the theory of Dirichlet's series into the foreground of interest, and this exceedingly competent presentation of the subject will help to create and sustain even more interest.

Algebraic Equations. By G. B. Mathews. No. 6 of the Cambridge Tracts in Mathematics and Mathematical Physics. Second Edition. Pp. viii + 64. 2s. 6d. net. 1915. (Cambridge: University Press.)

The first edition of this book was issued in 1907, and, as is well known, gave an excellent account of the theory of equations according to the ideas of Galois. In this second edition is inserted the condition that a general quintic may be metacyclic in the field of its coefficients. The discovery and calculation of it are due to Mr. W. E. H. Berwick. This is all that it is necessary to say about this new edition of a most valuable Tract.

Relativity. By A. W. Conway. No. 3 of the Edinburgh Mathematical Tracts. Pp. viii + 43. 2s. net. 1915. (London: G. Bell & Sons, Ltd.)

This little Tract contains four lectures on Einstein's deductions of fundamental relations; transformations of electromagnetic equations; applications to radiation and electron theory; and Minkowski's transformation. As many of the audience had their chief interests in other branches of mathematical science, it was necessary to start *ab initio*; and the subject is treated in the historical order and brought down to the stage in which it was left by Minkowski. The presentation is clear and good, and the historical method undoubtedly helps to rouse and keep up the reader's interest. Philip E. B. Jourdain.

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